Centres of Mass of Uniform Solids of Revolution

Solids of revolution are generated by rotating a curve in the xy plane around either the x- or y-axis. A **uniform** solid of revolution has constant density, and its centre of mass can be found using an integral formula. The y-coordinate of the centre of mass for a solid generated by rotation about the x-axis is zero since solids of revolution are symmetric about their axis of rotation. Solids generated by rotating a function $y = f(x)$ between the points $x = 0$ and $x = a$ have a centre of mass given by

$$\bar{x} = \frac{\int_0^a \pi x y^2 dx}{\int_0^a \pi y^2 dx} = \frac{\int_0^a \pi x f(x)^2 dx}{\int_0^a \pi f(x)^2 dx}.$$

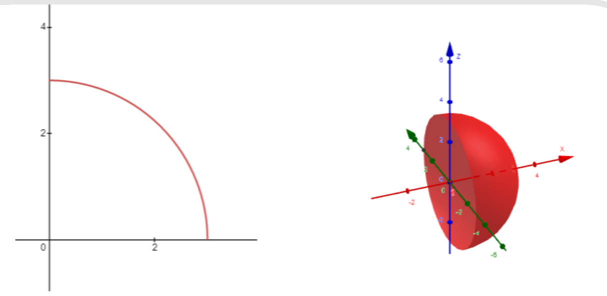
Notice that $\int_0^a \pi f(x)^2 dx$ is the volume of the solid of revolution. As uniform solids have uniform density, their mass is directly proportional to their volume, and so their volume can be used to calculate their centre of mass.

Example 1:

a) By considering the solid of revolution generated by rotating the curve $f(x) = \sqrt{r^2 - x^2}$ around the x-axis, find the position of the centre of mass of a hemisphere of radius r.

b) Hence, find the coordinates of the centre of mass of the solid of revolution generated by rotating the function $f(x) = \sqrt{289 - x^2}$ about the x-axis.

a) The function $f(x) = \sqrt{r^2 - x^2}$ is the equation of a circle of radius r centred at the origin $(x^2 + y^2 = r^2)$ rearranged for y in terms of x. Therefore, its rotation about the x- axis will generate a hemisphere when defined for $x \geq 0$. The plane curve shown on the right is $f(x)$ with $r = 3$. To the right of it is the corresponding solid of revolution, which is a hemisphere of radius 3.



Use the integral formula to find the x coordinate of the centre of mass. The y coordinate will be zero due to the solid being symmetric about the x axis. Cancel the constants π, and expand the integrands. Integrate both functions and simplify to find the final answer in terms of r. The position of the centre of mass will be directly above the centre of its base since hemispheres are symmetric about their vertical axis.

$$\bar{x} = \frac{\int_0^a \pi x f(x)^2 dx}{\int_0^a \pi f(x)^2 dx} = \frac{\int_0^r x\left(\sqrt{r^2 - x^2}\right)^2 dx}{\int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx}$$

$$= \frac{\int_0^r xr^2 - x^3\, dx}{\int_0^r r^2 - x^2\, dx} = \frac{\left[\frac{x^2r^2}{2} - \frac{x^4}{4}\right]_0^r}{\left[r^2x - \frac{x^3}{3}\right]_0^r}$$

$$= \frac{\frac{r^4}{2} - \frac{r^4}{4}}{r^3 - \frac{r^3}{3}} = \frac{\frac{r^4}{4}}{\frac{2r^3}{3}}$$

$$= \frac{3r}{8}$$

$\therefore$ The centre of mass of a hemisphere is $\frac{3r}{8}$ above the centre of its base since it is symmetrical about its vertical axis.

b) Comparing the function to $f(x) = \sqrt{r^2 - x^2}$, this solid of revolution is a hemisphere of radius $r = 17$. Substitute this value into the expression for $\bar{x}$, and use the fact that the y coordinate of the centre of mass is zero to get the coordinates for the hemisphere's centre of mass.

$$f(x) = \sqrt{289 - x^2} = \sqrt{17^2 - x^2}$$

The centre of mass lies on the x-axis, with y-coordinate

$$\frac{3r}{8} = \frac{3 \cdot 17}{8} = \frac{51}{8}$$

$\therefore$ the coordinates of the hemisphere's centre of mass are

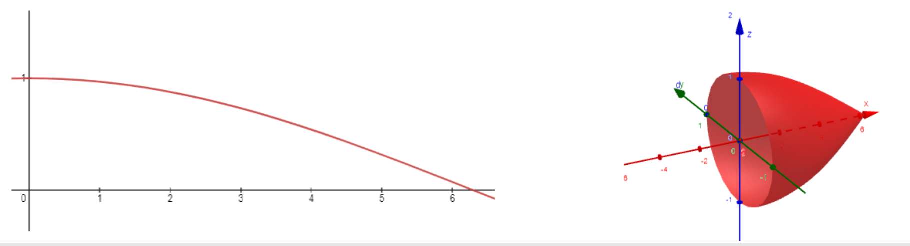
$$\begin{pmatrix} \frac{51}{8} \\ 0 \end{pmatrix}$$

This is a general result. Similar results for other solids can be derived in a similar way. The following table displays the results that are provided in the formula booklet. Note that a shell is a hollow solid.

Solid hemisphere of radius r	$\frac{3r}{8}$ above the centre of the hemisphere
Hemispherical shell of radius r	$\frac{r}{2}$ above the centre of the hemispherical shell
Solid cone or pyramid of height h	$\frac{h}{4}$ above the base, along the line from the centre of the base to the vertex
Conical shell of height h	$\frac{h}{3}$ above the base, along the line from the centre of the base to the vertex

Example 2: Find the centre of mass of the solid of revolution generated by rotating the curve $f(x) = \cos\left(\frac{x}{4}\right)$ between the points $x = 0$ and $x = 2\pi$ around the x-axis.

The plane curve $f(x) = \cos\left(\frac{x}{4}\right)$ along with its corresponding solid of revolution, generated via rotation about the x-axis.



Use the integral formula to find the x coordinate of the centre of mass. Evaluate the numerator and denominator individually, starting with the denominator. Use a double angle result to simplify the term $\cos^2(x)$, making it more straightforward to integrate. For the integral on the numerator, use integration by parts, differentiating x to arrive at a simpler integral. Once both the numerator and denominator are evaluated, take their quotient to arrive at the x coordinate of the centre of mass.

$$\bar{x} = \frac{\int_0^a \pi x f(x)^2 dx}{\int_0^a \pi f(x)^2 dx} = \frac{\int_0^{2\pi} x\cos^2\left(\frac{x}{4}\right) dx}{\int_0^{2\pi} \cos^2\left(\frac{x}{4}\right) dx}$$

Using the double angle formula $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$,

$$\int_0^{2\pi} \cos^2\left(\frac{x}{4}\right) dx = \frac{1}{2}\int_0^{2\pi} 1 + \cos\left(\frac{x}{2}\right) dx = \frac{1}{2}\left[x + 2\sin\left(\frac{x}{2}\right)\right]_0^{2\pi} = \pi$$

$$\int_0^{2\pi} x\cos^2\left(\frac{x}{4}\right) dx = \frac{1}{2}\int_0^{2\pi} x\left(1 + \cos\left(\frac{x}{2}\right)\right) dx = \frac{1}{2}\left(\int_0^{2\pi} x dx + \int_0^{2\pi} x\cos\left(\frac{x}{2}\right) dx\right)$$

Firstly, $\int_0^{2\pi} x dx = \left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi^2$. For $\int_0^{2\pi} x\cos\left(\frac{x}{2}\right) dx$, use integration by parts with

$$u = x, dv = \cos\left(\frac{x}{2}\right) \Rightarrow \frac{du}{dx} = 1, v = 2\sin\left(\frac{x}{2}\right)$$

$$\therefore \int_0^{2\pi} x\cos\left(\frac{x}{2}\right) dx = \left[2x\sin\left(\frac{x}{2}\right)\right]_0^{2\pi} - 2\int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[2x\sin\left(\frac{x}{2}\right) + 4\cos\left(\frac{x}{2}\right)\right]_0^{2\pi} = (-4 - 4) = -8$$

$$\therefore \int_0^{2\pi} x\cos^2\left(\frac{x}{4}\right) dx = \frac{1}{2}(2\pi^2 - 8) = \pi^2 - 4$$

$$\therefore \bar{x} = \frac{\pi^2 - 4}{\pi} = \pi - \frac{4}{\pi}$$

The solid of revolution is symmetric about the x axis, and so the y coordinate of the centre of mass is 0.

The centre of mass of this solid of revolution is

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \pi - \frac{4}{\pi} \\ 0 \end{pmatrix}$$